Heavy-Tailed Innovations in the R Package stochvol

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Abstract

We document how sampling from a conditional Student's t distribution is implemented in **stochvol**. Moreover, a simple example using EUR/CHF exchange rates illustrates how to use the augmented sampler. We conclude with results and implications.

Keywords: Student's t distribution, data augmentation, EUR/CHF exchange rates.

Preface

This note serves as a preliminary add-on to the more elaborate article "Dealing with Stochastic Volatility in Time Series using the R package **stochvol**" (Kastner 2016a). It discusses and relaxes the restriction to conditionally normal errors in the vanilla stochastic volatility (SV) model.

1. The SV model with Student's t errors

Several authors have suggested to use non-normal conditional innovation distributions for stochastic volatility modeling. Examples include the Student's t distribution (Harvey, Ruiz, and Shephard 1994), the extended Generalized Inverse Gaussian (Silva, Lopes, and Migon 2006), (semi-)parametric innovations (Jensen and Maheu 2010; Delatola and Griffin 2011), or the GH skew Student's t distribution (Nakajima and Omori 2012). In the following, we describe how the estimation of the SV model with Student's t errors is implemented in the R (R Core Team 2016) package **stochvol** (Kastner 2016b).

Let $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^\top$ be a vector of returns with mean zero. The SV model with Student's t errors (in short SV-t) is given through

$$
\begin{aligned}
y_t|h_t, \nu &\sim t_\nu(0, \exp h_t)\,, && (1)\\
h_t|h_{t-1}, \mu, \phi, \sigma_\eta &\sim \mathcal{N}\big(\mu + \phi(h_{t-1} - \mu), \sigma_\eta^2\big)\,, && (2)\\
h_0|\mu, \phi, \sigma_\eta &\sim \mathcal{N}\big(\mu, \sigma_\eta^2/(1 - \phi^2)\big)\,, && (3)
\end{aligned}
$$

i.e., conditionally on h_t, the data is assumed to follow a zero-mean non-standardized Student's t distribution with ν degrees of freedom and variance $(\nu \exp h_t)/(\nu - 2)$ for $\nu > 2$. Following Chib, Nardari, and Shephard (2002), we assume that a priori the degrees of freedom parameter

$\nu \sim \mathcal{U}(a, b)$, i.e., follows a uniform distribution with support on the real interval (a, b). All other prior components are chosen as in Kastner (2016a).

2. Usage

Estimating a stochastic volatility model with conditional t errors via **stochvol** is very similar to estimating a model with standard Gaussian errors, differing only through specifying a non-`NA` argument `priornu`. This triggers the sampler specified in Section 3. To provide an example, we investigate the historical daily EUR/CHF exchange rates and display these in Figure 1.

```
R> library(stochvol)
R> data(exrates)
R> par(mfrow = c(2, 1), mar = c(1.7, 1.7, 1.7, 0.1), mgp = c(1.6, 0.6, 0))
R> plot(exrates$date, exrates$CHF, type = 'l', main = 'Price of 1 EUR in CHF')
R> dat <- logret(exrates$CHF, demean = TRUE)
R> plot(exrates$date[-1], dat, type = 'l', main = 'Demeaned log returns')
```

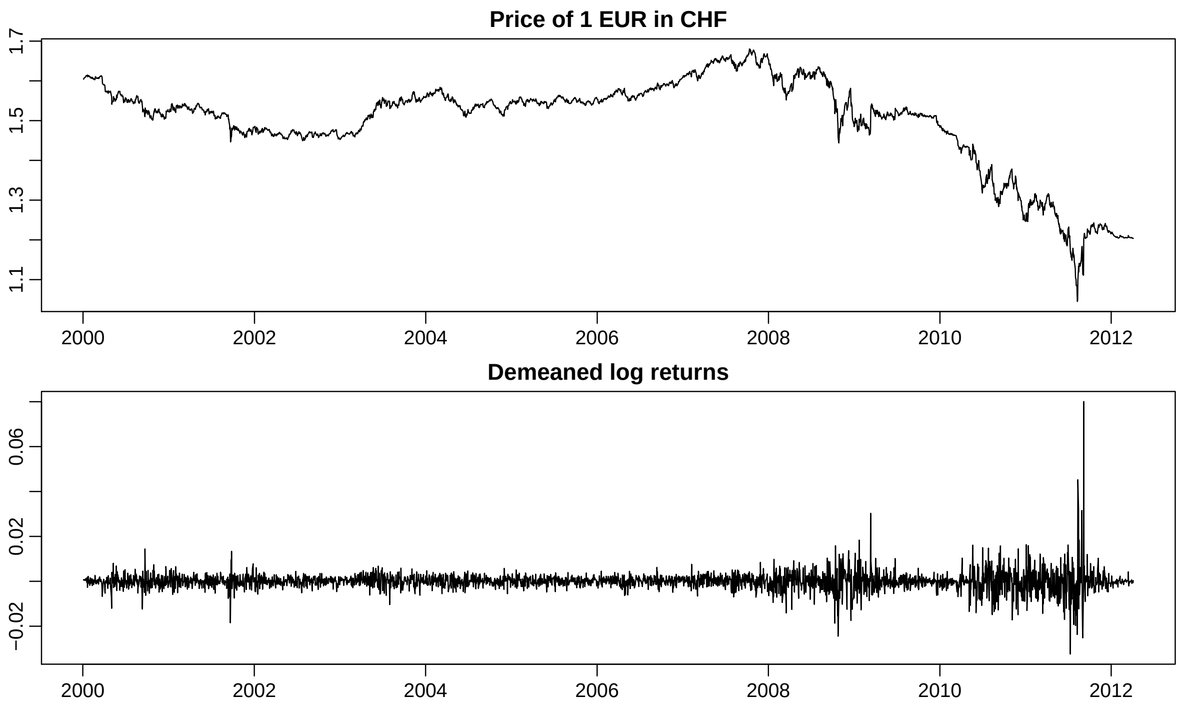

Figure 1: Levels and demeaned log returns of EUR in CHF.

By specifying the argument `priornu` (a two-element vector containing the lower and upper bounds of the uniform prior for ν), we can trigger the sampler to allow for heavy-tailed conditional innovations.

```
R> rest <- svsample(dat, priormu = c(-12, 1), priorphi = c(20, 1.1),
+    priorsigma = 0.1, priornu = c(2, 100), burnin = 2000)
R> plot(rest, showobs = FALSE)
```

Results are displayed in Figure 2, containing the output from the SV-t model. Row 1 depicts $\exp(h_t/2)$ for $t \in \{1, \dots, n\}$; row 2 shows the time varying standard deviations given through $\sqrt{\nu/(\nu-2)}\exp(h_t/2)$ for $t \in \{1, \dots, n\}$; row 3 portrays trace plots and row 4 outlines the corresponding smoothed kernel density estimates for the four parameters μ, ϕ, σ, and ν. It is worth noting that ν is estimated to lie between 6 and 18 with high posterior probability, indicating evidence for the presence of heavy tails even after catering for stochastic volatility. The extra flexibility of the SV-t sampler seems to allow for increased persistence ϕ and smaller variance of log-volatility σ^2, resulting in smoother time-varying volatility estimates.

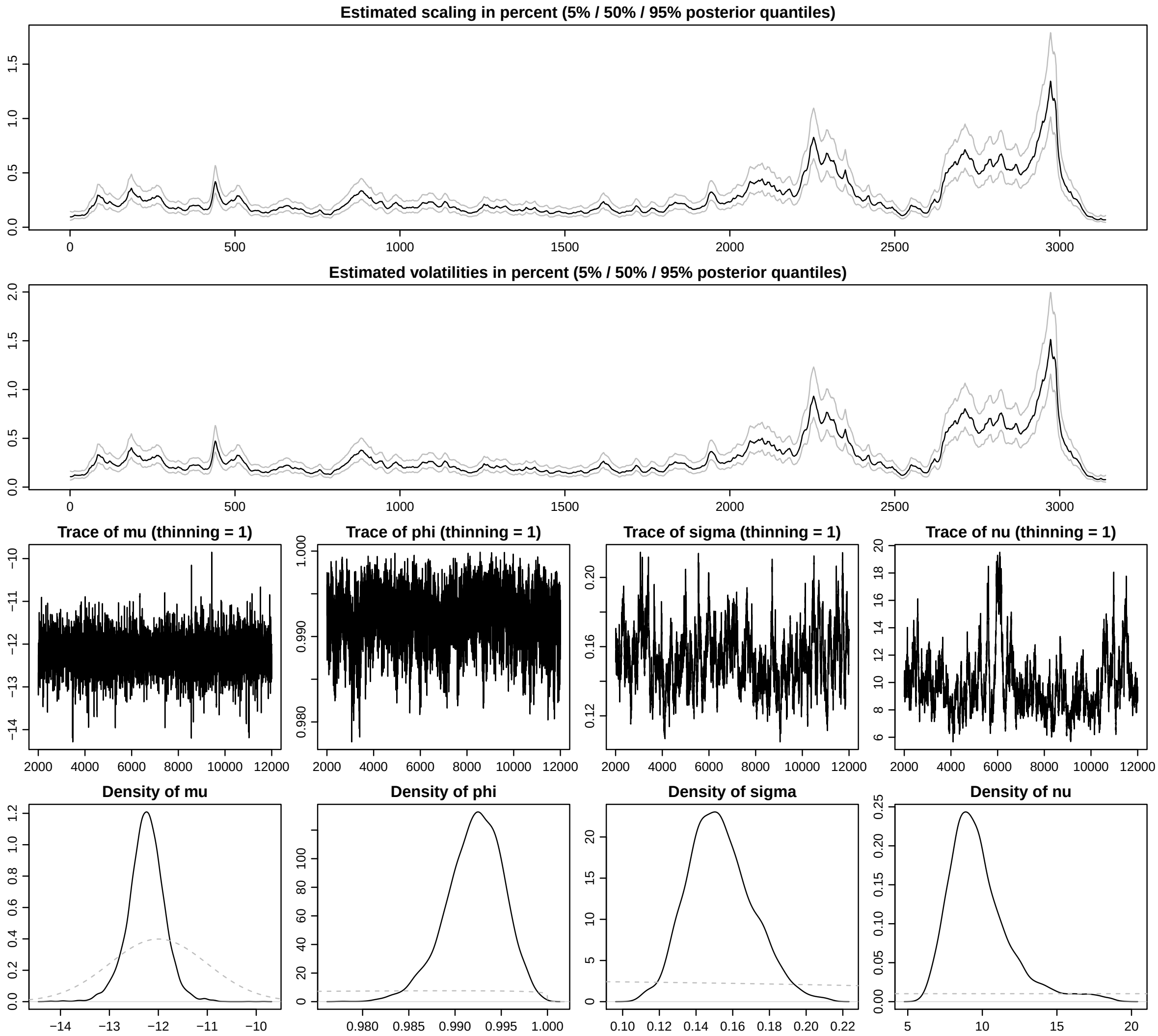

Figure 2: Standard output of the `plot` method when applied to an `svdraws` object containing posterior draws from an SV model with Student's t errors.

We investigate one-day-ahead out-of-sample predictive performance of an AR(1) model with (a) homoskedastic, (b) SV, (c) SV-t, (d) GARCH(1,1) errors, applied to the raw exchange rate data. Details about this procedure are provided in Chapter 5 of Kastner (2016a). The results, summarized in Figure 3, speak in favor of the SV-t model for this dataset.

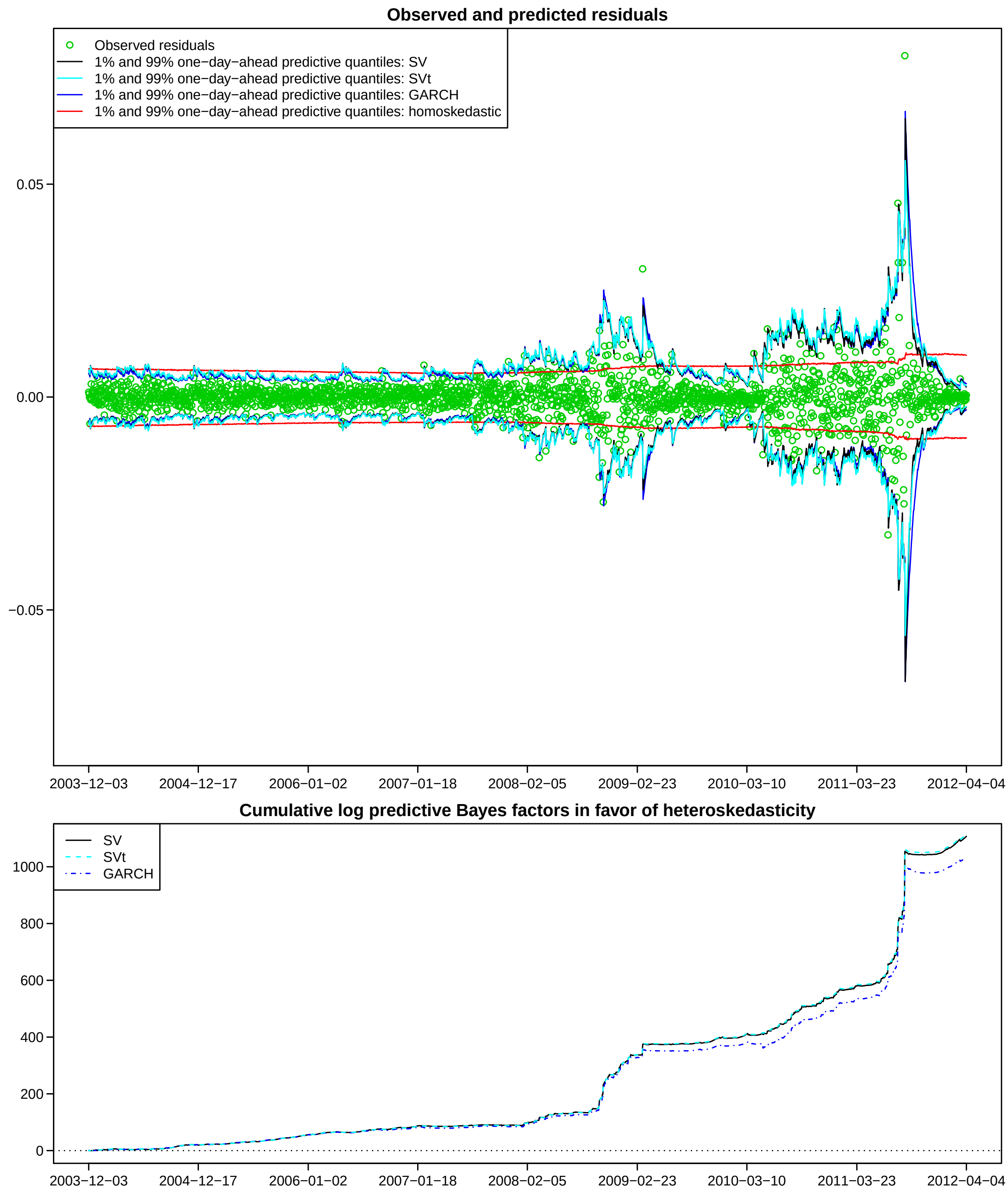

Figure 3: Log predictive one-day-ahead Bayes factors in favor of SV, SV-t, and GARCH errors over the homoskedastic model. The final log predictive Bayes factors aggregate to 1107.44 (SV), 1115.36 (SV-t), and 1033.53 (GARCH), respectively, thus providing strong evidence for the SV-t model.

3. Necessary modifications in the sampling scheme

The Student's t distribution appearing in Equation 1 can be conveniently expressed as a scale mixture of normal distributions,

$$\begin{aligned} y_t|h_t, \tau_t &\sim \mathcal{N}(0, \tau_t \exp h_t)\,, \\ \tau_t|\nu &\sim \mathcal{G}^{-1}(\nu/2, \nu/2)\,, \end{aligned}$$

where $\mathcal{N}\big(\mu, \sigma_\eta^2\big)$ denotes the normal distribution with mean μ and variance σ_η^2, and $\mathcal{G}^{-1}(a, b)$ denotes the inverse gamma distribution with shape and scale parameters a and b, respectively. Treating $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)^\top$ as latent data and letting $\tilde{y}_t = y_t/\sqrt{\tau_t}$ for $t \in \{1, \dots, n\}$, we have

$$\tilde{y}_t|h_t \sim \mathcal{N}(0, \exp h_t)\,,$$

and the AWOL sampler described in Kastner and Frühwirth-Schnatter (2014) can directly be applied to the transformed data. To obtain draws from the newly introduced variables $\boldsymbol{\tau}$ and ν, two additional steps are required.

3.1. Sampling the auxiliary variables

It is easy to see that the marginal posterior is given through

$$\tau_t|y_t, h_t, \nu \sim \mathcal{G}^{-1}\left(\frac{\nu+1}{2}, \frac{\nu + y_t^2 \exp(-h_t)}{2}\right),$$

independently for each $t \in \{1, \dots, n\}$. Obtaining draws from this distribution is straightforward.

3.2. Sampling the degrees of freedom parameter

The full conditional posterior of the degrees of freedom parameter, $\nu|\cdot$, only depends on $\boldsymbol{\tau}$. Its density is given through the product of n univariate truncated inverse gamma densities which may be written as

$$p(\nu|\cdot) = p(\nu|\boldsymbol{\tau}) \propto \left(\frac{\nu}{2}\right)^{n\nu/2} \Gamma\left(\frac{\nu}{2}\right)^{-n} \left(\prod_{t=1}^n \tau_t\right)^{-\nu/2} \exp\left\{-\frac{\nu}{2}\sum_{t=1}^n \frac{1}{\tau_t}\right\} \tag{4}$$

for $\nu \in (a, b)$ and zero elsewhere.

For obtaining draws from this distribution, we use an independence Metropolis-Hastings update. We follow Chib and Greenberg (1994), who introduced the idea of specifying an independence proposal through numerical maximization of the log-density. For the problem at hand, we consequently aim for optimizing

$$\log p(\nu|\boldsymbol{\tau}) = \frac{n\nu}{2}\log(\nu/2) - n\log\Gamma(\nu/2) - \frac{\nu}{2}\sum_{t=1}^n\left(\log\tau_t + \frac{1}{\tau_t}\right) + C, \tag{5}$$

with first and second derivatives given through

$$\frac{\partial \log p(\nu|\boldsymbol{\tau})}{\partial \nu} = \frac{n}{2}\Big(1 + \log(\nu/2) - \psi^{(0)}(\nu/2)\Big) - \frac{1}{2}\sum_{t=1}^n\left(\log\tau_t + \frac{1}{\tau_t}\right), \tag{6}$$

$$\frac{\partial^2 \log p(\nu|\boldsymbol{\tau})}{\partial \nu^2} = \frac{n}{2\nu} - \frac{n}{4}\psi^{(1)}(\nu/2), \tag{7}$$

where $\psi^{(m)}$ denotes the polygamma function of order m. Using the above, it is easy to numerically find

$$\begin{aligned} \hat{\nu} &= \arg\max_{\nu} \log p(\nu|\boldsymbol{\tau}), \\ B_{\hat{\nu}} &= -1 \Bigg/ \frac{\partial^2 \log p(\nu|\boldsymbol{\tau})}{\partial \nu^2}\Bigg|_{\nu=\hat{\nu}}, \end{aligned}$$

and a proposal candidate ν_{prop} may be drawn from a normal distribution with mean $\hat{\nu}$ and variance $B_{\hat{\nu}}$ (the Laplace approximation). Letting $\phi(x|\hat{\nu}, B_{\hat{\nu}})$ denote the corresponding density function, the acceptance probability is equal to $\min\{1, R\}$ with

$$R = \frac{p(\nu_{\text{prop}}|\boldsymbol{\tau})}{p(\nu_{\text{old}}|\boldsymbol{\tau})} \times \frac{\phi(\nu_{\text{old}}|\hat{\nu}, B_{\hat{\nu}})}{\phi(\nu_{\text{prop}}|\hat{\nu}, B_{\hat{\nu}})}.$$

4. Conclusion

We have shown how a simply data augmentation trick can be utilized to generalize the core sampler in **stochvol** in order to cater for potentially heavier-tailed innovation distributions. However, several caveats are called for:

- Even though the uniform prior for ν has been used widely, more robust alternatives are probably preferred, cf. Frühwirth-Schnatter and Pyne (2010) and the references therein.
- Sampling the degrees of freedom parameter ν conditionally on $\boldsymbol{\tau}$ can be very inefficient if ν becomes large (and thus the plain vanilla SV model suffices). We recommend to resort to the original SV sampler in this case.
- Leaving aside the additional computational burden, it is trivial to incorporate this extension into samplers employing **stochvol** as part of a larger MCMC scheme (e.g. Huber 2014; Kastner, Frühwirth-Schnatter, and Lopes 2014; Dovern, Feldkircher, and Huber 2015). Nevertheless, at the current stage of development, this should be conducted with caution by carefully investigating the convergence of the posterior draws.

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