Marshall-Oklin Extended Exponential (MOEE) distribution.

The Cumulative distribution function

For $\alpha > 0$ and $\lambda > 0$, the two-parameter MOEE distribution has the distribution function;

$$F(x;\alpha,\lambda) = 1 - \frac{\alpha}{e^{\lambda x} - (1-\alpha)} = \frac{1-e^{-\lambda x}}{1-(1-\alpha)\,e^{-\lambda x}} \quad ;(\alpha,\lambda) > 0,\ x > 0$$

Here α and λ are the tilt and scale parameters respectively.

The probability density function

Therefore, MOEE distribution has the density function

$$f(x;\alpha,\lambda) = \frac{\alpha\,\lambda\,e^{-\lambda x}}{\left\{1-(1-\alpha)\,e^{-\lambda x}\right\}^2} \quad ; \qquad (x > 0, \lambda > 0, \alpha > 0),$$

The density is log-convex, for $0 < \alpha < 1$, and log-concave, for $\alpha \geq 1$.

The Survival/Reliability function

The survival function

$$R(x;\alpha,\lambda) = \frac{\alpha}{e^{\lambda x} - (1-\alpha)} = \frac{\alpha\,e^{-\lambda x}}{1-(1-\alpha)\,e^{-\lambda x}} \quad ;(\alpha,\lambda) > 0,\ x > 0$$

The hazard function

The hazard rate

$$h(x;\alpha,\lambda) = \frac{\lambda}{1-(1-\alpha)e^{-\lambda x}} = \frac{\lambda e^{\lambda x}}{e^{\lambda x} - \bar{\alpha}} \qquad (x > 0, \lambda > 0, \alpha > 0)$$

Note that $h(x; 1, \lambda) = \lambda$, that $h(x; \alpha, \lambda)$ is decreasing in x for $0 < \alpha < 1$, and that $h(x; \alpha, \lambda)$ is increasing in x for $\alpha \geq 1$.

Indicators

The function $\log f(x; \alpha, \lambda)$ is convex, for $0 < \alpha < 1$, and concave, for $\alpha \geq 1$. This result can be verified by differentiating $\log f(x ; \alpha, \lambda)$ with respect to x. Of course, this means that for $\alpha < 1$, $f(x ; \alpha, \lambda)$ is decreasing, and for $\alpha \geq 1$, $f(x ; \alpha, \lambda)$ is unimodal.

By solving $d \log f(x ; \alpha, \lambda)/dx = 0$, it is readily verified that a random variable X with density f(x ; α, λ has the mode

$$\text{Mode} = \begin{cases} 0 & ; \quad \alpha \le 2 \\ \lambda^{-1} \log(\alpha - 1) & ; \quad \alpha > 2 \end{cases}$$

$$\text{Median} = \frac{1}{\lambda}\{\log(1+\alpha)\}$$

$$\text{Mean} = -\frac{\alpha \log \alpha}{\lambda(1-\alpha)}$$

It is easy to see that med(X), mode(X) and E(X) are all increasing in α and decreasing in the scale parameter λ. From the monotonicity of $\log x$ and the fact that $\log x \le x - 1$ $(x > 0)$, it follows that

$$\text{mode}(X) \le \text{median}(X) \le \alpha / \lambda \le \text{mean}(X).$$

The quantile function

$$x_q = \frac{1}{\lambda} \log\left\{1 + \frac{\alpha\, q}{(1-q)}\right\}$$

The random deviate

The random deviate can be generated by

$$x = \frac{1}{\lambda} \log\left\{1 + \frac{\alpha\, u}{(1-u)}\right\} \; ; 0 < u < 1$$

where u has the U(0, 1) distribution.

The log-density

$$\text{loglikelihood} = \log \alpha + \log \lambda - \lambda x - 2 \log\left\{1 - (1-\alpha)e^{-\lambda x}\right\}$$

The log-likelihood function

Let $\underline{x} = (x_1, \ldots, x_n)$ be a random sample of size n from MOEE(α, λ), then the log-likelihood function L(α, λ) can be written as;

$$L(\alpha, \lambda) = n \log \alpha + n \log \lambda + \lambda \sum_{i=1}^{n} x_i - 2 \sum_{i=1}^{n} \log\left\{1 - (1-\alpha)\, e^{-\lambda x_i}\right\}$$

The normal equations become;

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^{n} \frac{e^{-\lambda x_i}}{\left\{1 - (1-\alpha)\, e^{-\lambda x_i}\right\}} = 0$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i - 2\lambda \sum_{i=1}^{n} \frac{(1-\alpha)\, x_i\, e^{-\lambda x_i}}{\left\{1-(1-\alpha)\, e^{-\lambda x_i}\right\}} = 0$$

The cumulative hazard function

The cumulative hazard function H(x) defined as

$$H(x) = \int_0^x h(x)\, dx = -\log\{1 - F(x)\} = -\log\{R(x)\}$$

can be obtained with the help of pmoee()function by choosing arguments *lower.tail=FALSE* and *log.p=TRUE.* i.e.

- *pmoee(x, alpha, lambda, lower.tail=FALSE, log.p=TRUE)*

Failure rate average (fra) and Conditional survival function(crf)

Two other relevant functions useful in reliability analysis are failure rate average (fra) and conditional survival function(crf) The failure rate average of X is given by

$$FRA(x) = \frac{H(x)}{x} = \frac{-\log\{1-F(x)\}}{x} = \frac{-\log\{R(x)\}}{x} \qquad , x > 0,$$

where H(x) is the cumulative hazard function. An analysis for FRA(x) on x permits to obtain the IFRA and DFRA classes.

The survival function (s.f.) and the conditional survival of X are defined by

$$R(x) = 1 - F(x)$$

and $$R(x \mid t) = \frac{R(x+t)}{R(x)} \quad , t > 0, x > 0, R(\cdot) > 0,$$

respectively, where F(·) is the cdf of X. Similarly to h(x) and FRA(x), the distribution of X belongs to the new better than used (NBU), exponential, or new worse than used (NWU) classes, when $R(x \mid t) < R(x)$, $R(t \mid x) = R(x)$, or $R(x \mid t) > R(x)$, respectively.

References:

1. Marshall, A. W., Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Biometrika 84(3):641–652.
2. Marshall, A. W., Olkin, I.(2007). *Life Distributions: Structure of Nonparametric, Semiparametric, and Parametric Families.* Springer, New York.