POMDP: Introduction to Partially Observable Markov Decision Processes

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The R package **pomdp** provides an interface to 'pomdp-solve', a solver (written in C) for Partially Observable Markov Decision Processes (POMDP). The package enables the user to simply define all components of a POMDP model and solve the problem using several methods. The package also contains functions to analyze and visualize the POMDP solutions (e.g., the optimal policy).

In this document we will give a very brief introduction to the concept of POMDP, describe the features of the R package, and illustrate the usage with a toy example.

Introduction on POMDPs

A partially observable Markov decision process (POMDP) is a combination of an MDP to model system dynamics with a hidden Markov model that connects unobservant system states to observations. The agent can perform actions which affect the system (i.e., may cause the system state to change) with the goal to maximize a reward that depends on the sequence of system state and the agent's actions. However, the agent cannot directly observe the system state, but at each discrete point in time, the agent makes observations that depend on the state. The agent uses these observations to form a belief of in what state the system currently is. This belief is called a belief state and is expressed as a probability distribution over the states. The solution of the POMDP is a policy prescribing which action is optimal for each belief state.

The POMDP framework is general enough to model a variety of real-world sequential decision-making problems. Applications include robot navigation problems, machine maintenance, and planning under uncertainty in general. The general framework of Markov decision processes with incomplete information was described by Karl Johan Åström in 1965 in the case of a discrete state space, and it was further studied in the operations research community where the acronym POMDP was coined. It was later adapted for problems in artificial intelligence and automated planning by Leslie P. Kaelbling and Michael L. Littman (Littman 2009).

A discrete-time POMDP can formally be described as a 7-tuple

$$\mathcal{P} = (S, A, T, R, \Omega, O, \lambda),$$

where

- $S = \{s_1, s_2, \dots, s_n\}$ is a set of states,
- $A = \{a_1, a_2, \dots, a_m\}$ is a set of actions,
- T is a set of conditional transition probabilities $T(s' \mid s, a)$ for the state transition $s \to s'$.
- $R : S \times A \to \mathbb{R}$ is the reward function,
- $\Omega = \{o_1, o_2, \dots, o_k\}$ is a set of observations,
- O is a set of conditional observation probabilities $O(o \mid s', a)$, and
- $\gamma \in [0, 1]$ is the discount factor.

At each time period, the environment is in some state $s \in S$. The agent chooses an action $a \in A$, which causes the environment to transition to state $s' \in S$ with probability $T(s' \mid s, a)$. At the same time, the agent receives an observation $o \in \Omega$ which depends on the new state of the environment with probability $O(o \mid s', a)$. Finally, the agent receives a reward $R(s, a)$. Then the process repeats. The goal is for the agent to choose actions at each time step that maximizes its expected future discounted reward, i.e., she chooses the actions at each time t that

$$\max E\left[\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t)\right]$$

.

For a finite time horizon, only the expectation over the sum up to the time horizon is used.

Package Functionality

Solving a POMDP problem with the **pomdp** package consists of two steps:

1. Define a POMDP problem using the function `POMDP`, and

2. solve the problem using `solve_POMDP`.

Defining a POMDP Problem

The `POMDP` function has the following arguments, each corresponds to one of the elements of a POMDP.

```
str(args(POMDP))
```

```
## function (discount = 0, states, actions, observations, transition_prob,
##     observation_prob, reward, start = "uniform", max = TRUE, name = NA)
```

where

- `discount` is the discount factor γ in range $[0, 1]$,
- `states` defines the set of states S,
- `actions` defines the set of actions A,
- `observations` defines the set of observations Ω,
- `start` is the initial probability distribution over the system states S,
- `transition_prob` defines the conditional transition probabilities $T(s' \mid s, a)$,
- `observation_prob` specifies the conditional observation probabilities $O(o \mid s', a)$,
- `reward` specifies the reward function R,
- `values` indicates whether the problem is a maximization (`values = "reward"`, the default) or a minimization (`values = "cost"`), and
- `name` used to give the POMDP problem a name.

While specifying the discount rate and the set of states, observations and actions is straight-forward. Some arguments can be specified in different ways. The initial belief state `start` can be specified as

- A vector of n probabilities in $[0, 1]$, that add up to 1, where n is the number of states.

```
r   start = c(0.5 , 0.3 , 0.2)
```

- The string ' "uniform" ' for a uniform distribution over all states.

```
r   start = "uniform"
```

- A vector of integer indices specifying a subset as start states. The initial probability is uniform over these states. For example, only state 3 or state 1 and 3:

```
r   start = 3   start = c(1, 3)
```

- A vector of strings specifying a subset as start states.

`r start <- "state3" start <- c("state1" , "state3")`

- A vector of strings starting with `"-"` specifying which states to exclude from the uniform initial probability distribution.

`r start = c("-" , "state2")`

The transition probabilities, `transition_prob`, depend on the `end.state` s', the `start.state` s and the `action` a. The set of conditional transition probabilities can be specified in several ways:

- A data frame with 4 columns, where the columns specify `action`, `start.state`, `end.state` and the `probability`, respectively. Actions and states are either identified by their names or their index number. An asterisk (`*`) can be used as a placeholder for all possible actions or states. To make creating the data frame easier the helper function `T_` is provides.

`r transition_prob = rbind( T_("action1", "state1", "state2", 0.1), T_("action2", "state1", "state3", 0.9), T_("*" , "state2", "*", 1) )` * A named list of m matrices where each matrix represents one of the m actions. Each matrix is square of size $n \times n$ where n is the number of states. Matrices can also be defined as `"identity"` or `"uniform"`.

"'r transition_prob = list("action1" = matrix(c(0.1, 0.4, 0.5, 0, 0.7, 0.3, 0.4, 0.4, 0.2), nrow = 3 , byrow = TRUE) , "action2" = matrix(c(0, 0.6, 0.4, 0.1, 0.9, 0, 0.7, 0.3, 0), nrow = 3 , byrow = TRUE))

transition_prob = list("action1" = matrix(c(0.1, 0.4, 0.5, 0, 0.7, 0.3, 0.4, 0.4, 0.2), nrow = 3 , byrow = TRUE) , "action2" = "uniform") "'

The observation probabilities, `observation_prob`, depend on the `action`, the `end.state`, and the `observation`. The set of conditional observation probabilities can be specified in several ways:

- A data frame with 4 columns, where the columns specify `action`, `end.state`, `observation` and the `probability`, respectively. The first 3 columns could be either the name or the index of the action, state, or observation. The special character `*` can be used to indicate that the probability applies for all actions, states or observations. The helper function `O_` is provided to specify the data frame.

`r observation_prob = rbind( O_("*", "state1", "obs1", 0.1), O_("*", "state1", "obs2", 0.9), O_("*", "state2", "obs1", 0.3), O_("*", "state2", "obs2", 0.7), O_("*", "state3", "obs1", 0.5), O_("*", "state3", "obs2", 0.6) )` * A named list of m matrices, one for each actions. Each matrix is of size $n \times o$ where n is the number of states and o is the number of observations. (each matrix should have a name in the list and the name should be one of the actions). Matrices can also be defined as `"identity"` or `"uniform"`.

"'r observation_prob = list("action1" = matrix(c(0.1, 0.9, 0.3, 0.7, 0.4, 0.6), nrow = 3, byrow = TRUE) , "action2" = matrix(c(0.1, 0.9, 0.3, 0.7, 0.4, 0.6), nrow = 3, byrow = TRUE))

observation_prob = list("action1" = "uniform", "action2" = matrix(c(0.1, 0.9, 0.3, 0.7, 0.4, 0.6), nrow = 3, byrow = TRUE)) "'

The reward function, `reward`, depends on `action`, `start.state`, `end.state` and the `observation`. The reward function can be specified several ways:

- A data frame with 5 columns, where the columns specify `action`, `start.state`, `end.state`, `observation` and the reward, respectively. The first 4 columns could be either the name or the index of the action, state, or observation. The special character `*` can be used to indicate that the same reward applies for all actions, states or observations. The helper function `R_` is provided to specify the data frame.

`r reward = rbind( R_("action1", "*", "state1", "*", 10000), R_("action1", "*", "state2", "*", 2000), R_("action2", "*", "state1", "*", 50), R_("action2", "*", "state2", "*", 100) )` * A named list of m lists, where m is the number of actions (names should be the actions). Each list contains n named matrices where each matrix is of size $n \times o$, in which n is the number of states and o is the number of observations. Names of these matrices should be the name of states.

```
r  reward = list(    "action1" = list(       "state1" = matrix(c(1, 2, 3, 4, 5, 6) ,
nrow = 3 , byrow = TRUE),        "state2" = matrix(c(3, 4, 5, 2, 3, 7) , nrow = 3 ,
byrow = TRUE),        "state3" = matrix(c(6, 4, 8, 2, 9, 4) , nrow = 3 , byrow = TRUE)),
"action2" = list(       "state1" = matrix(c(3, 2, 4, 7, 4, 8) , nrow = 3 , byrow = TRUE),
"state2" = matrix(c(0, 9, 8, 2, 5, 4) , nrow = 3 , byrow = TRUE),        "state3" =
matrix(c(4, 3, 4, 4, 5, 6) , nrow = 3 , byrow = TRUE)))
```

Solving a POMDP

POMDP problems are solved with the function `solve_POMDP` with the following arguments.

```
str(args(solve_POMDP))
```

```
## function (model, horizon = NULL, method = "grid", parameter = NULL,
##     verbose = FALSE)
```

The `model` argument is a POMDP problem created using the `POMDP` function, but it can also be the name of a POMDP file using the format described on www.pomdp.org. The `horizon` argument specifies the finite time horizon (i.e, the number of time steps) considered in solving the problem. If the horizon is unspecified (i.e., `NULL`), then the algorithm continues running iterations till it converges to the infinite horizon solution (Anthony Rocco Cassandra 1998). The `method` argument specifies what algorithm the solver should use. Available methods including `"grid"`, `"enum"`, `"twopass"`, `"witness"`, and `"incprune"`. Further solver parameters can be specified as a list as `parameters`. The list of available parameters can be obtained using the function `solve_POMDP_parameter()`. Finally, `verbose` is a logical that indicates whether the solver output should be shown in the R console or not. The output of this function is an object of class POMDP.

Helper Functions

The package offers several functions to help with managing POMDP problems and solutions.

The functions `model`, `solution`, and `solver_output` extract different elements from a POMDP object returned by `solve_POMDP()`.

The package provides a plot function to visualize the solution's policy graph using the package **igraph**. The graph itself can be extracted from the solution using the function `policy_graph()`.

The Tiger Problem Example

We will demonstrate how to use the package with the Tiger Problem (Anthony R. Cassandra, Kaelbling, and Littman 1994).

A tiger is put with equal probability behind one of two doors, while treasure is put behind the other one. You are standing in front of the two closed doors and need to decide which one to open. If you open the door with the tiger, you will get hurt by the tiger (negative reward), but if you open the door with the treasure, you receive a positive reward. Instead of opening a door right away, you also have the option to wait and listen for tiger noises. But listening is neither free nor entirely accurate. You might hear the tiger behind the left door while it is actually behind the right door and vice versa.

The states of the system are the tiger behind the left door (tiger-left) and the tiger behind the right door (tiger-right).

Available actions are: open the left door (open-left), open the right door (open-right) or to listen (listen).

Rewards associated with these actions depend on the resulting state: +10 for opening the correct door (the door with treasure), -100 for opening the door with the tiger. A reward of -1 is the cost of listening.

As a result of listening, there are two observations: either you hear the tiger on the right (tiger-right), or you hear it on the left (tiger-left).

The transition probability matrix for the action listening is identity, i.e., the position of the tiger does not change. Opening either door means that the game restarts by placing the tiger uniformly behind one of the doors.

Specifying the Tiger Problem

The problem can be specified using function `POMDP()` as follows.

```
library("pomdp")

TigerProblem <- POMDP(
  name = "Tiger Problem",

  discount = 0.75,

  states = c("tiger-left" , "tiger-right"),
  actions = c("listen", "open-left", "open-right"),
  observations = c("tiger-left", "tiger-right"),

  start = "uniform",

  transition_prob = list(
    "listen" = "identity",
    "open-left" = "uniform",
    "open-right" = "uniform"),

  observation_prob = list(
    "listen" = matrix(c(0.85, 0.15, 0.15, 0.85), nrow = 2, byrow = TRUE),
    "open-left" = "uniform",
    "open-right" = "uniform"),

  reward = rbind(
    R_("listen",     "*",           "*", "*", -1  ),
    R_("open-left",  "tiger-left",  "*", "*", -100),
    R_("open-left",  "tiger-right", "*", "*", 10  ),
    R_("open-right", "tiger-left",  "*", "*", 10  ),
    R_("open-right", "tiger-right", "*", "*", -100)
  )
)

TigerProblem
```

```
## Unsolved POMDP model: Tiger Problem
```

Note that we use for each component the way that lets us specify them in the easiest way (i.e., for observations and transitions a list and for rewards a data frame created with the `R_` function).

Solving the Tiger Problem

Now, we can solve the problem using the default algorithm. We use the finite grid method which implements a form of point-based value iteration that can find approximate solutions also for difficult problems.

```
tiger_solved <- solve_POMDP(TigerProblem)
tiger_solved
```

```
## Solved POMDP model: Tiger Problem
```

```
##      solution method: grid
##      policy graph nodes: 5
##      total expected reward:1.93343898266653
```

The output is an object of class POMDP which contains the solution.

```r
solution(tiger_solved)
```

```
## POMDP solution
##
## $method
## [1] "grid"
##
## $parameter
## NULL
##
## $total_expected_reward
## [1] 1.933439
##
## $initial_pg_node
## [1] 3
##
## $belief_states
##         tiger-left  tiger-right node
##  [1,] 5.000000e-01 5.000000e-01    3
##  [2,] 8.500000e-01 1.500000e-01    4
##  [3,] 1.500000e-01 8.500000e-01    2
##  [4,] 9.697987e-01 3.020134e-02    5
##  [5,] 3.020134e-02 9.697987e-01    1
##  [6,] 9.945344e-01 5.465587e-03    5
##  [7,] 5.465587e-03 9.945344e-01    1
##  [8,] 9.990311e-01 9.688763e-04    5
##  [9,] 9.688763e-04 9.990311e-01    1
## [10,] 9.998289e-01 1.711147e-04    5
## [11,] 1.711147e-04 9.998289e-01    1
## [12,] 9.999698e-01 3.020097e-05    5
## [13,] 3.020097e-05 9.999698e-01    1
## [14,] 9.999947e-01 5.329715e-06    5
## [15,] 5.329715e-06 9.999947e-01    1
## [16,] 9.999991e-01 9.405421e-07    5
## [17,] 9.405421e-07 9.999991e-01    1
## [18,] 9.999998e-01 1.659782e-07    5
## [19,] 1.659782e-07 9.999998e-01    1
## [20,] 1.000000e+00 2.929027e-08    5
## [21,] 2.929027e-08 1.000000e+00    1
## [22,] 1.000000e+00 5.168871e-09    5
## [23,] 5.168871e-09 1.000000e+00    1
## [24,] 1.000000e+00 9.121536e-10    5
## [25,] 9.121536e-10 1.000000e+00    1
##
## $pg
##   node     action tiger-left tiger-right
## 1    1  open-left          3           3
## 2    2     listen          3           1
## 3    3     listen          4           2
```

```
## 4    4     listen          5           3
## 5    5 open-right          3           3
##
## $alpha
##          coef_1     coef_2
## [1,] -98.549921  11.450079
## [2,] -10.854299   6.516937
## [3,]   1.933439   1.933439
## [4,]   6.516937 -10.854299
## [5,]  11.450079 -98.549921
##
## $belief_proportions
##       tiger-left tiger-right
## [1,] 0.003349418 0.996650582
## [2,] 0.150000000 0.850000000
## [3,] 0.500000000 0.500000000
## [4,] 0.850000000 0.150000000
## [5,] 0.996650582 0.003349418
```

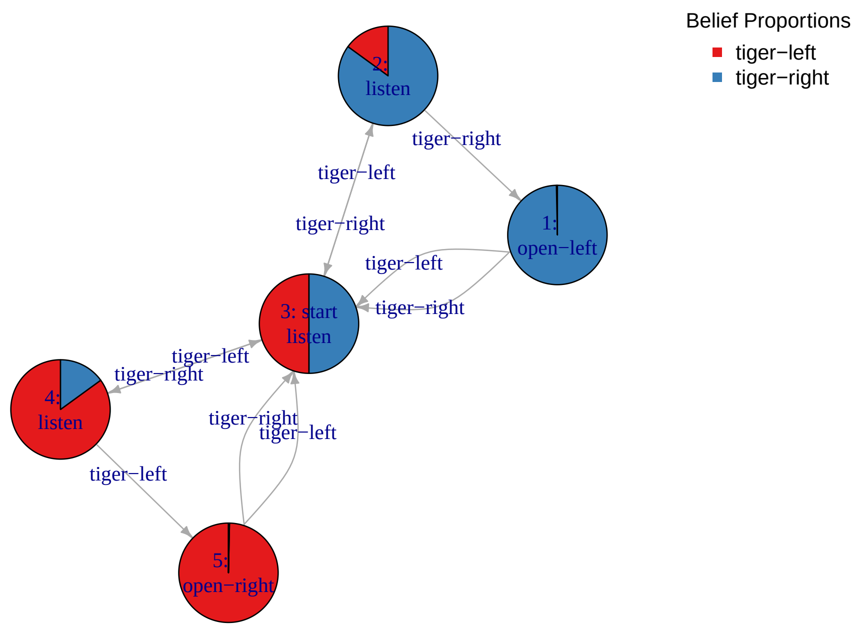
The solution contains the following elements:

- **`total_expected_reward`:** The total expected reward of the optimal solution.
- **`initial_belief_state`:** The index of the node in the policy graph that represents the initial belief state.
- **`belief_states`:** A data frame of all the belief states (rows) used while solving the problem. There is a column at the end that indicates which node in the policy graph is associated with the belief state. That is which segment in the value function (specified in **alpha** below) provides the best value for the given belief state.
- **`pg`:** A data frame containing the optimal policy graph. Rows are nodes in the graph are segements in the value function and each represents one or more belief states. Column two indicates the optimal action for the node. Columns three and after represent the transitions to new nodes in the policy graph depending on the next observation.
- **`alpha`:** A data frame with the coefficients of the optimal hyperplanes for the value function.
- **`belief_proportions`:** A data frame which summarized the belief states associated with a node in the policy graph by averaging the beliefs.

Visualization

In this section, we will visualize the policy graph provided in the solution by the `solve_POMDP` function.

```
plot(tiger_solved)
```



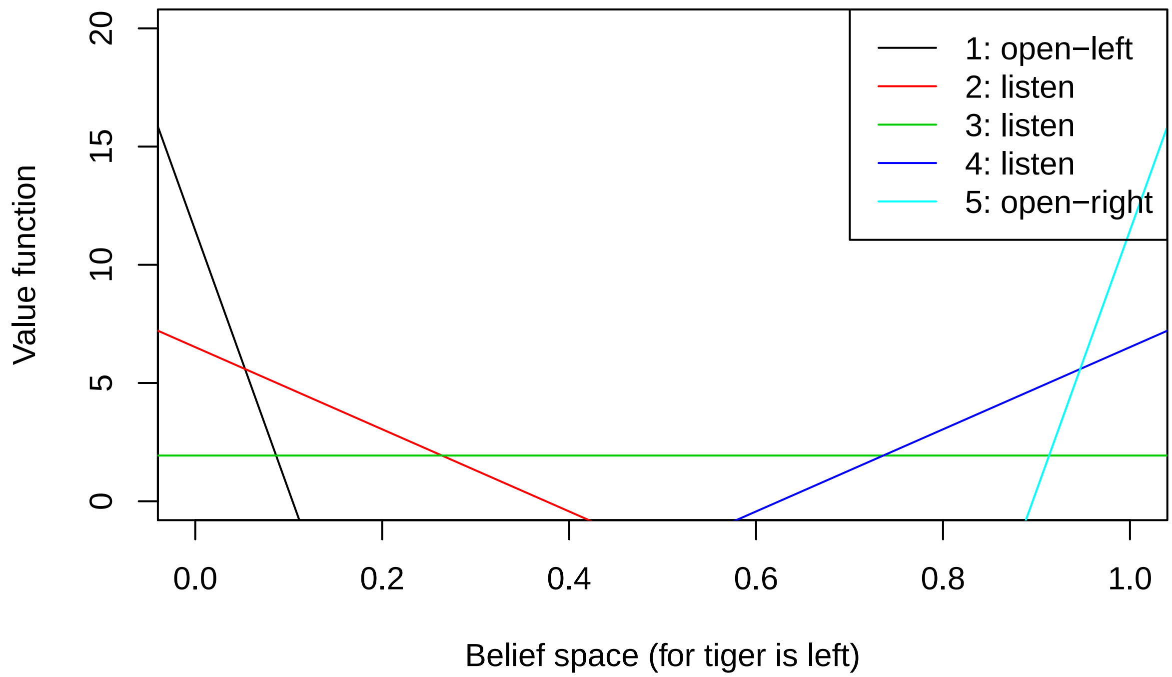
The policy graph can be easily interpreted. Without prior information, the agent starts at the node marked with "initial." In this case the agent beliefs that there is a 50-50 chance that the tiger is behind ether door. The optimal action is displayed inside the state and in this case is to listen. The observations are labels on the arcs. Let us assume that the observation is "tiger-left", then the agent follows the appropriate arc and ends in a node representing a belief (one ore more belief states) that has a very high probability of the tiger being left. However, the optimal action is still to listen. If the agent again hears the tiger on the left then it ends up in a note that has a close to 100% belief that the tiger is to the ledt and `open-right` is the optimal action. The are arcs back from the nodes with the open actions to the initial state reset the problem.

Since we only have two states, we can visualize the piecewise linear convex value function as a simple plot.

```
alpha <- solution(tiger_solved)$alpha
alpha
```

```
##          coef_1     coef_2
## [1,] -98.549921  11.450079
## [2,] -10.854299   6.516937
## [3,]   1.933439   1.933439
## [4,]   6.516937 -10.854299
## [5,]  11.450079 -98.549921
```

```
plot(NA, xlim = c(0, 1), ylim = c(0, 20), xlab = "Belief space (for tiger is left)",
  ylab = "Value function")
for(i in 1:nrow(alpha)) abline(a = alpha[i,2], b = alpha[i,1]- alpha[i,2], col = i, xpd = FALSE)
legend("topright", legend =
    paste0(1:nrow(alpha),": ", solution(tiger_solved)$pg[,"action"]), col = 1:nrow(alpha), lwd=1)
```



The lines represent the nodes in the policy graph and the optimal actions are shown in the legend.

References

Cassandra, Anthony R., Leslie Pack Kaelbling, and Michael L. Littman. 1994. "Acting Optimally in Partially Observable Stochastic Domains." In *Proceedings of the Twelfth National Conference on Artificial Intelligence*. Seattle, WA.

Cassandra, Anthony Rocco. 1998. "Exact and Approximate Algorithms for Partially Observable Markov Decision Processes." PhD thesis, Providence, RI, USA: Brown University.

Littman, Michael L. 2009. "A Tutorial on Partially Observable Markov Decision Processes." *Journal of Mathematical Psychology* 53 (3): 119–25. doi:10.1016/j.jmp.2009.01.005.