phull: p-hull in R

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Outline

1. p-hull and its properties
2. Examples
3. Computation

Preliminaries

Given an arbitrary $0 < p < \infty$, $x_0, y_0 \in \mathbb{R}$, $a \geq 0$ and $b \geq 0$, let

$$E_{p,a,b}^{(x_0,y_0)} = \left\{ (x,y) \in \mathbb{R}^2 : \left| \frac{y - y_0}{b} \right|^p + \left| \frac{x - x_0}{a} \right|^p \leq 1 \right\}. \quad (1)$$

Moreover, for $p = \infty$ we have

$$E_{p,a,b}^{(x_0,y_0)} = \left\{ (x,y) \in \mathbb{R}^2 : \max \left\{ \left| \frac{y - y_0}{b} \right|, \left| \frac{x - x_0}{a} \right| \right\} \leq 1 \right\}, \quad (2)$$

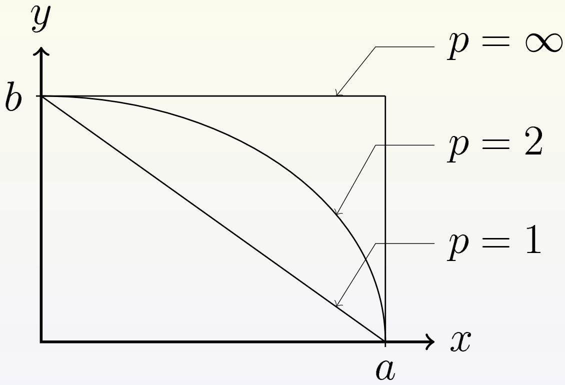
and for $p = 0$

$$E_{p,a,b}^{(x_0,y_0)} = \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{cccc} & x \in [x_0 - a, x_0 + a] & \wedge & y = y_0 \\ \vee & y \in [y_0 - b, y_0 + b] & \wedge & x = x_0 \end{array} \right\}. \quad (3)$$

We call $E_{p,a,b}^{(x_0,y_0)}$ the p-**ellipse of size** (a, b) **centered at** (x_0, y_0).

Preliminaries

Illustration: $\partial E_{p,a,b}^{(0,0)} \cap \mathbb{R}_0^+ \times \mathbb{R}_0^+$.



Preliminaries

We are given a finite planar set $Q = \{q_1, q_2, \ldots, q_n\}$, such that $q_i = (x_i, y_i) \in \mathbb{R}^2$, $i = 1, \ldots, n$ $(n \geq 4)$.

Let

$$\begin{aligned} x_{\mathrm{l}} &= \min_{p_i \in P} x_i, \\ x_{\mathrm{r}} &= \max_{p_i \in P} x_i, \\ y_{\mathrm{b}} &= \min_{p_i \in P} y_i, \\ y_{\mathrm{t}} &= \max_{p_i \in P} y_i. \end{aligned}$$

Then $B(Q) = [x_{\mathrm{l}}, x_{\mathrm{r}}] \times [y_{\mathrm{t}}, y_{\mathrm{b}}]$ is the **minimal bounding rectangle** of Q.

Preliminaries

For a fixed $p \geq 0$ let

$$\begin{aligned}
C_p^{\mathrm{bl}}(Q) &= \bigcup\nolimits_{a,b:\ Q \notin \operatorname{int} E_{p,a,b}^{(x_\mathrm{l},y_\mathrm{b})}} E_{p,a,b}^{(x_\mathrm{l},y_\mathrm{b})}, \\
C_p^{\mathrm{br}}(Q) &= \bigcup\nolimits_{a,b:\ Q \notin \operatorname{int} E_{p,a,b}^{(x_\mathrm{r},y_\mathrm{b})}} E_{p,a,b}^{(x_\mathrm{r},y_\mathrm{b})}, \\
C_p^{\mathrm{tr}}(Q) &= \bigcup\nolimits_{a,b:\ Q \notin \operatorname{int} E_{p,a,b}^{(x_\mathrm{r},y_\mathrm{t})}} E_{p,a,b}^{(x_\mathrm{r},y_\mathrm{t})}, \\
C_p^{\mathrm{tl}}(Q) &= \bigcup\nolimits_{a,b:\ Q \notin \operatorname{int} E_{p,a,b}^{(x_\mathrm{l},y_\mathrm{t})}} E_{p,a,b}^{(x_\mathrm{l},y_\mathrm{t})}.
\end{aligned}$$

We further on assume $\operatorname{int} C_p^{\mathrm{bl}}(Q)$, $\operatorname{int} C_p^{\mathrm{br}}(Q)$, $\operatorname{int} C_p^{\mathrm{tr}}(Q)$, $\operatorname{int} C_p^{\mathrm{tl}}(Q)$ are mutually exclusive.

p-hull

Definition

Let $Q = \{q_1, q_2, \ldots, q_n\} \subset \mathbb{R}^2$ and $p \geq 0$. The p-**hull** of Q, denoted by $H_p(Q)$, is defined by

$$H_p(Q) = \partial \left(B(Q) \setminus C_p^{\mathrm{bl}}(Q) \setminus C_p^{\mathrm{br}}(Q) \setminus C_p^{\mathrm{tr}}(Q) \setminus C_p^{\mathrm{tl}}(Q) \right). \qquad (4)$$

Properties of a p-hull

Proposition

Let $Q = \{q_1, q_2, \ldots, q_n\} \subset \mathbb{R}^2$ and $p \geq 0$. Then we have the following.

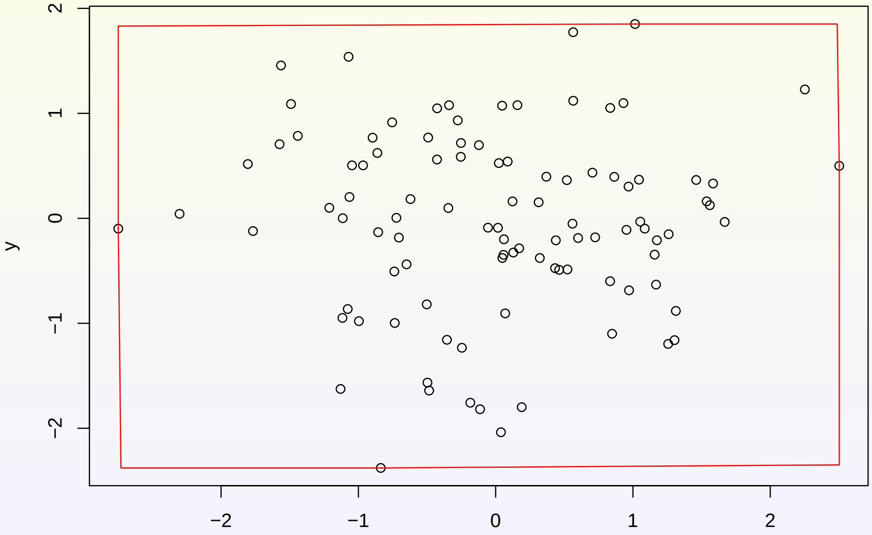
1. *If $p = 1$ then $H_p(Q)$ is the convex hull of Q.*
2. *If $p = \infty$ then $H_p(Q)$ is the X-Y hull of Q (see Nicholl et al, 1983).*
3. *If $p = 0$ then $H_p(Q) = \partial B(Q)$.*

Properties of a p-hull

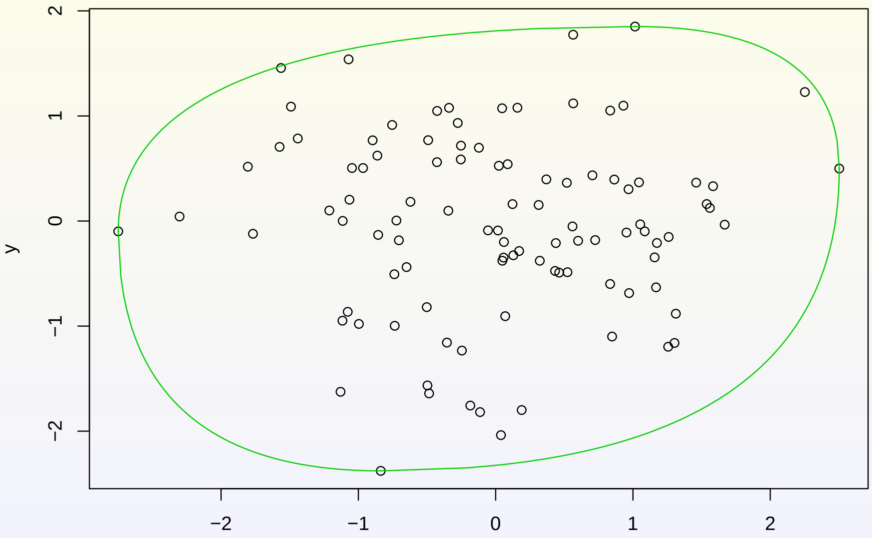
Other properties:

1. $H_p(Q)$ is translation- and scale-invariant for any $p \geq 0$, i.e. $\circledast H_p(Q) = H_p(\circledast Q)$.
2. $H_p(Q)$ is not rotation-invariant (thus it is orientation-dependent) for $p \neq 1$.
3. $H_p(Q)$ is convex for $p \leq 1$.
4. If $p' \geq p$, then $H_{p'}(Q) \subseteq H_p(Q)$.

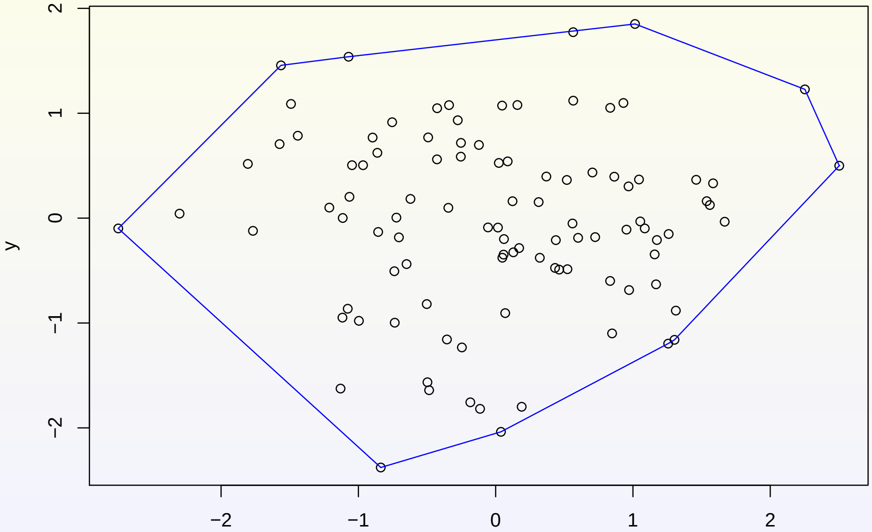
Example: $p = 0.1$



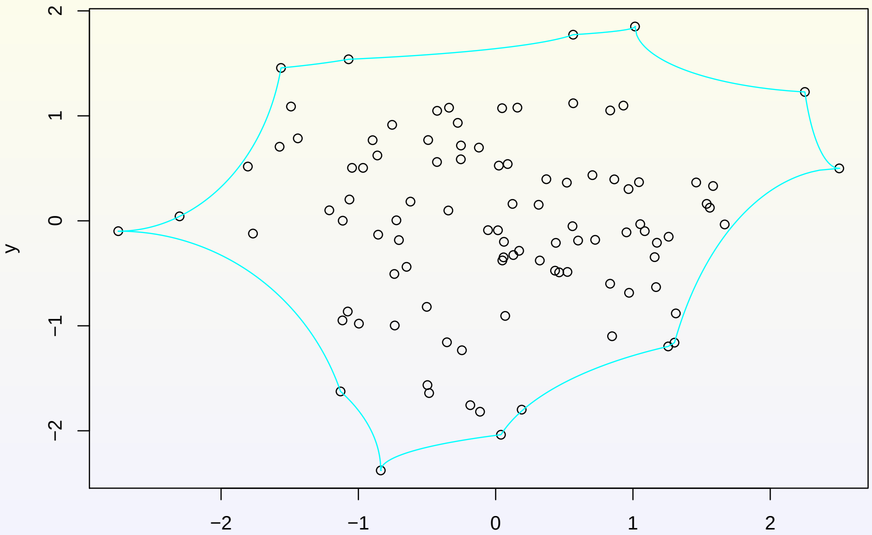
Example: $p = 0.5$



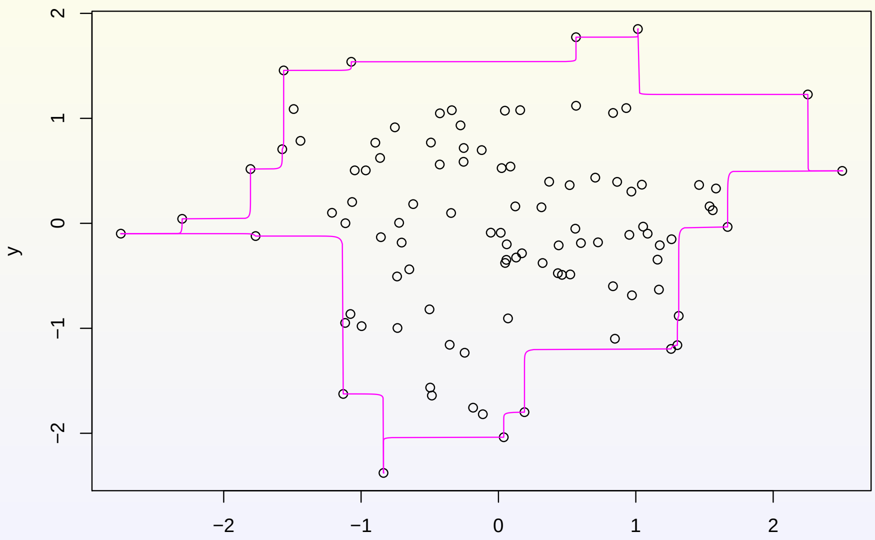
Example: $p = 1.0$



Example: $p = 2.0$



Example: $p = 50$



Applications

Among possible applications are:

1. Scientometrics: calculation of so-called L_p indices (Gągolewski, Grzegorzewski, 2009),
2. p.d.f. support estimation: given a random sample of points from distribution given by unknown p.d.f. $f(x, y)$, estimate $\operatorname{supp} f$,
3. ...

Computation

Let

$$
\begin{aligned}
q_{\mathrm{bl}_1} &= \mathop{\arg\min}_{q_i \in Q\colon\ x_i = x_\mathrm{l}} y_i, &\qquad q_{\mathrm{bl}_2} &= \mathop{\arg\min}_{q_i \in Q\colon\ y_i = y_\mathrm{b}} x_i,\\
q_{\mathrm{br}_1} &= \mathop{\arg\max}_{q_i \in Q\colon\ y_i = y_\mathrm{b}} x_i, &\qquad q_{\mathrm{br}_2} &= \mathop{\arg\min}_{q_i \in Q\colon\ x_i = x_\mathrm{r}} y_i,\\
q_{\mathrm{tr}_1} &= \mathop{\arg\max}_{q_i \in Q\colon\ x_i = x_\mathrm{r}} y_i, &\qquad q_{\mathrm{tr}_2} &= \mathop{\arg\max}_{q_i \in Q\colon\ y_i = y_\mathrm{t}} x_i,\\
q_{\mathrm{tl}_1} &= \mathop{\arg\min}_{q_i \in Q\colon\ y_i = y_\mathrm{t}} x_i, &\qquad q_{\mathrm{tl}_2} &= \mathop{\arg\max}_{q_i \in Q\colon\ x_i = x_\mathrm{l}} y_i.
\end{aligned}
$$

Note that all the points $\in \partial B(Q)$.

Computation (cont'd)

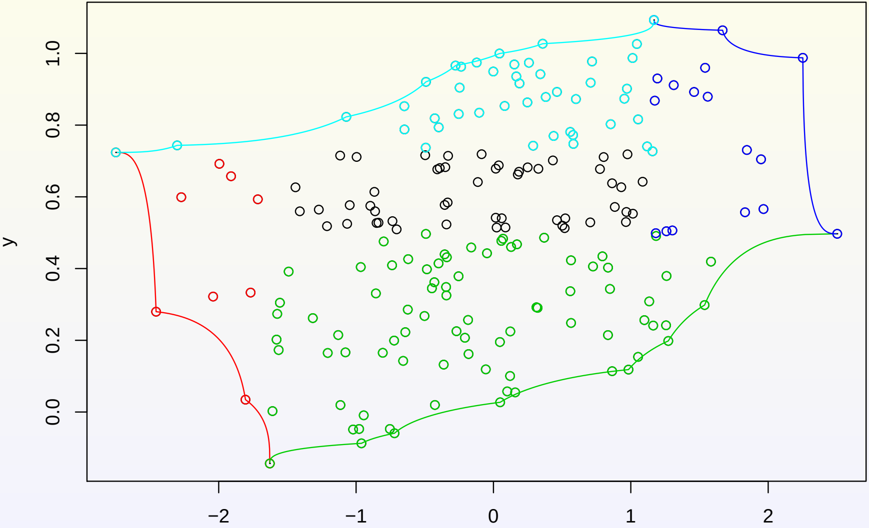
Decomposition:

$$\begin{aligned} H_p(Q) &= \partial\left(B(Q) \setminus C_p^{\mathrm{bl}}(Q) \setminus C_p^{\mathrm{br}}(Q) \setminus C_p^{\mathrm{tr}}(Q) \setminus C_p^{\mathrm{tl}}(Q)\right) \\ &= \left(\partial C_p^{\mathrm{bl}}(Q) \cup \partial C_p^{\mathrm{br}}(Q) \cup \partial C_p^{\mathrm{tr}}(Q) \cup \partial C_p^{\mathrm{tl}}(Q)\right) \cap B(Q) \\ &\cup\ \overline{q_{\mathrm{bl}_2} q_{\mathrm{br}_1}} \cup \overline{q_{\mathrm{br}_2} q_{\mathrm{tr}_1}} \cup \overline{q_{\mathrm{tr}_2} q_{\mathrm{tl}_1}} \cup \overline{q_{\mathrm{tl}_2} q_{\mathrm{bl}_1}}. \end{aligned} \tag{5}$$

Moreover:

$$\begin{aligned} \partial C_p^{\mathrm{bl}}(Q) &= \partial C_p^{\mathrm{bl}}(\{q_i \in Q :\ x_i \le x_{\mathrm{bl}_2} \wedge y_i \le y_{\mathrm{bl}_1}\}), \\ \partial C_p^{\mathrm{br}}(Q) &= \partial C_p^{\mathrm{br}}(\{q_i \in Q :\ x_i \ge x_{\mathrm{br}_1} \wedge y_i \le y_{\mathrm{br}_2}\}), \\ \partial C_p^{\mathrm{tr}}(Q) &= \partial C_p^{\mathrm{tr}}(\{q_i \in Q :\ x_i \ge x_{\mathrm{tr}_2} \wedge y_i \ge y_{\mathrm{tr}_1}\}), \\ \partial C_p^{\mathrm{tl}}(Q) &= \partial C_p^{\mathrm{tl}}(\{q_i \in Q :\ x_i \le x_{\mathrm{tl}_1} \wedge y_i \ge y_{\mathrm{tl}_2}\}). \end{aligned} \tag{6}$$

Computation (cont'd)



Computation (cont'd)

1. Naïve algorithm: $O(n^3)$ time. :-(
2. Algorithm proposed by Gągolewski, Nowakiewicz, Dębski (2009) — generalizes Graham's scan (Graham, 1972): $O(n \log n)$ time, $O(n)$ memory.
3. Output: $P \cap \partial C_p^{\mathrm{bl}}(Q)$ (without loss of generality).
4. Input: $p \geq 0$, $W = \{q_i \in Q : \ x_i \leq x_{\mathrm{bl}_2} \wedge y_i \leq y_{\mathrm{bl}_1}\}$ as an array **sorted** by x coordinate $w[1], \ldots, w[m]$.
5. Denotation: by $E_{p,q_i,q_j}^{(x_0,y_0)}$ we mean an p-ellipse centered at (x_0, y_0) interpolating $q_i \neq q_j$.

Computation (cont'd)

Create an empty stack $\mathbf{S}$;
Push $w[1]$ into $\mathbf{S}$;
$i := 2$;
while $(i < n)$ **and** $(w[i]_y \geq w[1]_y)$ **do**
    $i := i + 1$;
Push $w[i]$ into $\mathbf{S}$;
for $j = i + 1, i + 2, \ldots, n$ **do**
    if $(\mathbf{S}[\#\mathbf{S}]_y < w[j]_y)$ **then** {
        while $(\#\mathbf{S} \geq 2)$ **and** $(\mathbf{S}[\#\mathbf{S} - 1] \in E^{(x_{\mathrm{bl}}, y_{\mathrm{bl}})}_{p, \mathbf{S}[\#\mathbf{S}], w[j]})$ **do**
            Pop from $\mathbf{S}$;
        Push $w[j]$ into $\mathbf{S}$;
    }
return $\mathbf{S}$;

Computation (cont'd)

Implementation: `phull` 0.1-2 — package available on CRAN.
(http://cran.r-project.org/web/packages/phull/index.html)

Example: axes rotation.

```
library(phull); # load the library

translateAndRotate <- function(data, x0, y0, angle)
{ ... }

rotateAndTranslate <- function(data, x0, y0, angle)
{ ... }
```

Computation (cont'd)

```
set.seed(98765); n <- 1000; p <- 3.0;
data <- matrix(c(rnorm(n), rt(n, 10)), ncol=2); # input data
nres <- 50; # "resolution"

ptest <- phull(data, p=p);                       # compute the p-hull
discr_0 <- as.matrix(ptest, nres=nres);          # sample

print(ptest)
```

```
        p-hull, p=3

data: data
1000 points, bounding rectangle: (...)
```

Computation (cont'd)

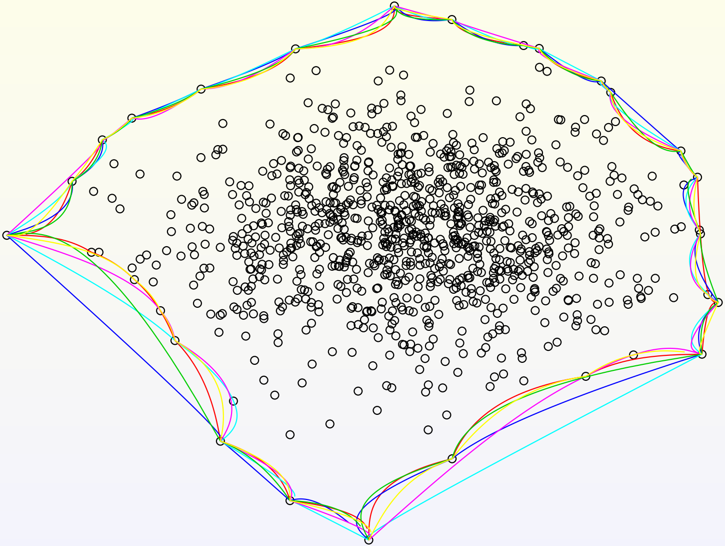
```
data2 <- translateAndRotate(data, angle=-pi/6
   -ptest$xrange[1], -ptest$yrange[1]);
ptest2 <- phull(data2, p=p);                     # compute the p-hull
discr_30 <- as.matrix(ptest2, nres=nres);  # sample
discr_30 <- rotateAndTranslate(discr_30, angle=pi/6,
   ptest$xrange[1], ptest$yrange[1]);

plot(data, type="p", pch=1);
lines(discr_0,  col=2);
lines(discr_30, col=4);
```

…and so on…

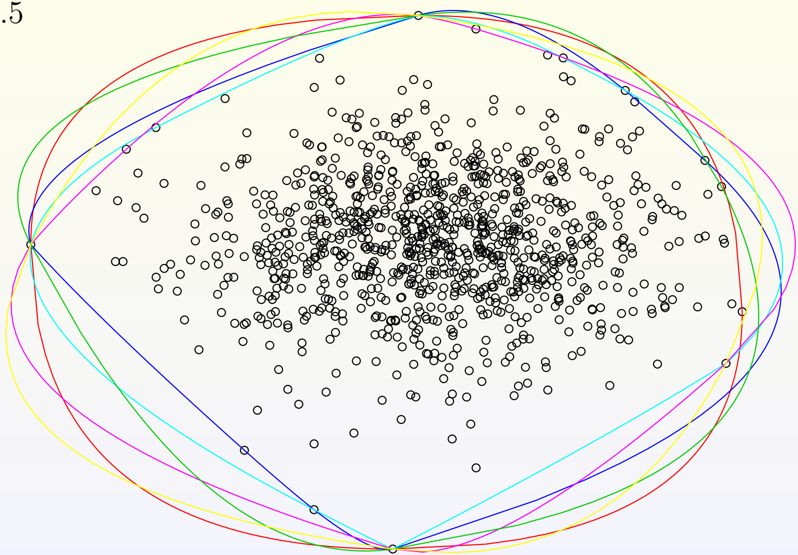
Computation (cont'd)

$p = 3$



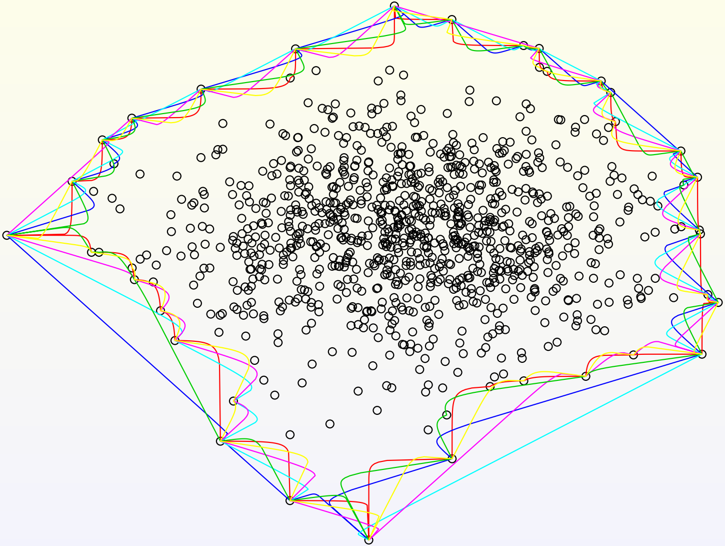
Computation (cont'd)

$p = 0.5$



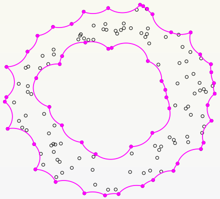
Computation (cont'd)

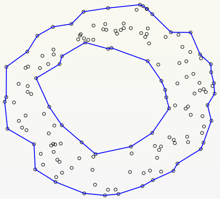
$p = 20$

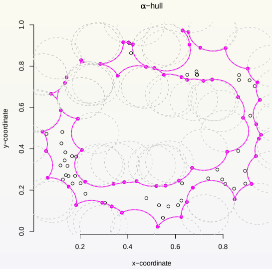


Related packages

alphahull (Pateiro-Lopez, Rodriguez-Casal, 2009): α-shapes (Edelsbrunner et al, 1983).







References

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- B. Pateiro-Lopez, A. Rodriguez-Casal (2009). *alphahull: Generalization of the convex hull of a sample of points in the plane*, see `alphahull` @ CRAN.

Thank you for your attention.